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DETERMINATION OF STRESS INTENSITY FACTORS FOR CRACKS OF COMPLEX SHAPE IN ANISOTROPIC PLATES

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The application of analytical methods to the problem of fatigue crack propagation and branching is complicated by the shortage of information on the stress distribution near the tip of cracks of complex configuration. A discussion of this problem and a survey of the studies in this area can be found in [1], for example. Below we develop a method of solving a problem concerning a system of cracks of complex form in an anisotropic half-plane. An efficient algorithm for numerical solution of the problem is proposed. A study is made of the effect of anisotropy of the material, the free edge of the plate, and the curvature of the crack on the stress intensity factors at the tips of the cracks.

1. We will examine an elastic plate made of a homogeneous anisotropic material occupying the region D = {x > 0}. The plate is weakened by smooth, curved, non-intersecting internal notches L_j ($j = 1, \ldots, k$), and is subjected to a system of external forces. We will assume that the edges of the notches are free of loads and are not in contact with one another.

The stresses in the plate are expressed through two analytic functions [2]:

$$(\sigma_x, \tau_{xy}, \sigma_y) = 2\,\mathrm{Re}\left\{\sum_{\nu=1}^{2} (\mu_\nu^2, -\mu_\nu, 1)\Phi_\nu(z_\nu)\right\}, \quad z_\nu = x + \mu_\nu y \quad (\nu = 1, 2), \tag{1.1}$$

where μ_ν are the roots of the characteristic equation.

We seek the unknown functions $\Phi_\nu(z_\nu)$ in the form

$$\Phi_\nu(z_\nu) = \Phi_\nu^0(z_\nu) + \Phi_\nu^1(z_\nu),$$
$$\Phi_\nu^1(z_\nu) = \frac{1}{2\pi i}\int_L \left\{\frac{\omega_\nu(\tau)\,d\tau_\nu}{\tau_\nu - z_\nu} - \frac{l_\nu s_\nu \overline{\omega_1(\tau)}\,d\overline{\tau}_1}{\overline{\tau}_1 - s_\nu z_\nu} - \frac{n_\nu m_\nu \overline{\omega_2(\tau)}\,d\overline{\tau}_2}{\overline{\tau}_2 - m_\nu z_\nu}\right\}. \tag{1.2}$$

Henceforth, we use the notation in [3]; $\Phi_\nu^0(z_\nu)$ is the solution for the half-plane without notches from the prescribed system of external forces. The values of $\Phi_\nu^0(z_\nu)$ will be assumed to be known. The functions $\Phi_\nu^1(z_\nu)$ were given in another form in [4].

The functions $\Phi_\nu(z_\nu)$ determined by Eqs. (1.2) satisfy the prescribed system of external forces, including the boundary conditions on the edge of the plate x = 0 and at infinity.

Inserting the limiting values of $\Phi_\nu(z_\nu)$ from (1.2) into the boundary conditions for L and parameterizing the contours $L_j = \{t = \tau^j(\xi);\ |\xi| < 1\}$, we obtain the following system of singular integral equations of the problem [3] to determine the unknown complex functions $\omega_\nu(t) = \{\omega_{\nu j}(t) | t \in L_j;\ j = 1, \ldots, k\}$:

$$\int_{-1}^{1} \frac{F_j(\xi, \eta)\,\chi_j(\eta)\,d\eta}{\eta - \xi} + \sum_{s=1}^{k}\int_{-1}^{1} \{k_1^{js}(\xi, \eta)\,\chi_s(\eta) + k_2^{js}(\xi, \eta)\,\overline{\chi_s(\eta)}\}\,d\eta = f_j(\xi),$$

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$$\omega_{1j}(t)=\chi_j(\xi),$$
$$\omega_{2j}(t)=-a(t)\omega_{1j}(t)-b(t)\overline{\omega_{1j}(t)},$$
$$k_1^{js}(\xi,\eta)=\frac{\dot{\tau}_1(\eta)}{2}\frac{d}{dt_1}\left\{\ln\frac{(\tau_1-t_1)(\bar{\tau}_2-m_1t_1)(\tau_1-\bar{s}_2\bar{t}_2)}{(\bar{\tau}_2-\bar{t}_2)}-\left|\frac{a_0}{b_0}\right|^2\ln\frac{(\tau_1-\bar{s}_2\bar{t}_2)(\tau_2-\bar{m}_1\bar{t}_1)}{(\tau_1-\bar{s}_1\bar{t}_1)(\tau_2-\bar{m}_2\bar{t}_2)}+\frac{1-\delta_{js}}{\tau_1-t_1}\dot{\tau}_1(\eta)\right\},$$
$$k_2^{js}(\xi,\eta)=\frac{\overline{a_0\dot{\tau}_1(\eta)}}{2\bar{b}_0}\frac{d}{dt_1}\left\{\ln\frac{(\bar{\tau}_1-\bar{t}_1)(\bar{\tau}_2-m_1t_1)(\tau_2-\bar{m}_2\bar{t}_2)}{(\bar{\tau}_2-\bar{t}_2)(\tau_2-\bar{m}_1\bar{t}_1)(\bar{\tau}_1-s_1t_1)}\right\}, \tag{1.3}$$
$$f_j(\xi)=\frac{-\pi i}{\overline{b(t)}}\{\overline{a(t)}\,\overline{\Phi_1^0(t_1)}+b(t)\Phi_1^0(t_1)+\overline{\Phi_2^0(t_2)}\},$$
$$F_j(\xi,\eta)=\frac{(\eta-\xi)\dot{\tau}_1^j(\eta)}{\tau_1^j(\eta)-t_1},$$
$$a_0=\frac{\mu_1-\bar{\mu}_1}{\mu_2-\bar{\mu}_2},\ b_0=\frac{\bar{\mu}_1-\bar{\mu}_2}{\mu_2-\bar{\mu}_2},\ t=\tau^j(\xi)\in L_j,\ \tau=\tau^s(\eta)\in L_s,$$

where δ_{js} is the Kronecker symbol; $\dot{\tau}=d\tau/d\eta$.

Equations (1.3), together with auxiliary conditions for the nonambiguity of the displacements in circumventing L_j [3]

$$\int_{-1}^{1}\chi_j(\eta)\dot{\tau}_1^j(\eta)\,d\eta=0\quad(j=1,\ldots,k) \tag{1.4}$$

completely determine the sought solution of the problem.

Let the plate be weakened by a single rectilinear notch $L=\{t=\tau(\xi)=x_0+le^{i\alpha}\xi;\ \ |\xi|<1\}$ $(0\leqslant\alpha<2\pi)$. Assuming $\chi(\xi)=i\varphi(\xi)/\varepsilon$ $(\varepsilon=\mu_1-\mu_2)$ and changing to an isotropic medium ($\varepsilon\to0$), we obtain an integral equation in the problem of a rectilinear notch in an isotropic half-plane

$$\int_{-1}^{1}\frac{\varphi(\eta)\,d\eta}{\eta-\xi}+\int_{-1}^{1}\{k_1(\xi,\eta)\varphi(\eta)+k_2(\xi,\eta)\overline{\varphi(\eta)}\}\,d\eta=f(\xi),$$
$$k_1(\xi,\eta)=\frac{1}{2}\frac{d}{d\xi}\ln(C^2+T^2)-2X(C+iT)^{-4}\{2iY(C+iT)\cos\psi+$$
$$+(\eta-\xi)C\sin^2\psi+T^2\sin\psi+2iT[(\eta+\xi)\sin\psi-Y]\sin\psi\}, \tag{1.5}$$
$$k_2(\xi,\eta)=4\frac{e^{-i\psi}CTX}{(C^2+T^2)^2},\ C=\frac{2x_0}{l}-(\eta+\xi)\sin\psi,$$
$$T=(\eta-\xi)\sin\psi,\ X=\frac{x_0}{l}-\eta\sin\psi,\ Y=\frac{x_0}{l}-\xi\sin\psi,\ \psi=\alpha-\frac{\pi}{2}.$$

This result agrees with the relations found in [1], for example.

2. Since the smooth internal notches L_j are non-intersecting, the kernels $k_p^{js}(\xi,\eta)$ in Eqs. (1.3) are continuous. The index of the system of singular integral equations (1.3) is equal to +1. The solution of the system, with auxiliary conditions (1.4) in the class of functions

$$\chi_j(\xi)=\chi_j^0(\xi)(1-\xi^2)^{-1/2} \tag{2.1}$$

[$\chi_j^0(\xi)$ are bounded functions which are continuous in accordance with Hölder's condition] exists and is unique [5].

Normally we reduce the solution of system (1.3), together with auxiliary conditions (1.4), to the solution of a system of linear algebraic equations

$$\frac{\pi}{N}\sum_{n=1}^{N}\left\{\frac{F_j(x_p,t_n)}{t_n-x_p}\chi_{jn}^0+\sum_{s=1}^{k}\left[k_{1,pn}^{js}\chi_{sn}^0+k_{2,pn}^{js}\bar{\chi}_{sn}^0\right]\right\}=f_{jp},$$
$$\frac{\pi}{N}\sum_{n=1}^{N}\dot{\tau}_1^j(t_n)\chi_{jn}^0=0,\ j=1,\ldots,k,\ p=1,\ldots,N-1, \tag{2.2}$$
$$k_{1,pn}^{js}=k_1^{js}(x_p,t_n),\ k_{2,pn}^{js}=k_2^{js}(x_p,t_n),\ x_p=\cos\frac{\pi}{N}p,\ t_n=\cos\frac{2n-1}{2N}\pi$$

for approximate values of the sought functions at Chebyshev nodes $\chi_{jn}^0 = \chi_j^0(t_n)$. An evaluation of the convergence of the solution of system (2.2) on the solutions of Eqs. (1.3) and (1.4) was given in [6], for example.

Having determined the following values from (2.2) and (1.3)

$$\omega_{\nu j}^0(\pm 1) = \lim_{\xi \to \pm 1} \omega_{\nu j}\left[\tau^j(\xi)\right](1-\xi^2)^{1/2}$$

and using asymptotic formulas in the neighborhood $c = \tau^j(\pm 1)$ of the ends of the notch L_j [3]

$$\Phi_\nu(z_\nu) \approx 2^{-3/2}\omega_{\nu j}^0(\pm 1)\{\mp \tau_1^j(\pm 1)/(z_\nu - c_\nu)\}^{1/2},$$

we can use Eqs. (1.1) to then find the asymptotic stress distribution at the crack tips.

In the case where the end a_p of the notch L_p is at the edge of the plate x = 0 or in the event of the intersection of notches at the point a_p, condition (1.4) is no longer satisfied at j = p and should be discarded. Fixed singularities appear in the kernels $k_\nu^{jp}(\xi, \eta)$ of integral equations (1.3), while the function $\omega_{1p}(t)$ will have a singularity at the point $t = a_p$ which differs from the root singularity. Its character is determined from integral equations of the problem (1.3) by the method in [7]. The numerical method of solution (2.2) is not valid in this case.

Below we use the simplified method of solution in [1] for boundary and branching cracks. As before, we seek the unknown functions $\chi_p(\xi)$ in the form (2.1), but instead of condition (1.4) at j = p we impose the condition $\chi_p^0(-1) = 0$ [$a_p = \tau^p(-1)$ — the branching point or the point at which the notch L_p reaches the edge of the plate]. This simplified method of solution is effective only when it is not necessary to determine the stress distribution in the neighborhood of the inflection point $a_p = \tau^p(-1)$. If it is necessary to study the stress distribution near the tip a_p, then we need to seek the solution in a form which properly reflects the singularity at the inflection point and to use more complicated formulas of integration.

3. Representations (1.2) and the algorithm for numerical solution of integral equations (1.3) prove to be an effective means of determining stresses in the neighborhood of the tips of notches of complex form in anisotropic and isotropic plates.

Presented below are results of calculations which are important in fracture mechanics in regard to the stress intensity factors for normal and shear stresses at the tip of a curved crack or a branching crack with straight branches in a half-plane:

$$K_1 = \lim_{t \to c} \sigma_n \sqrt{\frac{r}{l}},\quad K_2 = \lim_{t \to c} \tau_n \sqrt{\frac{r}{l}}.$$

Here, ℓ is a certain nominal dimension; $r = |t - c|$; c is the tip of the crack; t is a point lying on a tangent to the crack drawn through the tip c.

Calculations were performed for plates of isotropic and anisotropic materials with different degrees of anisotropy: a) $E = 27.61 \cdot 10^4$ MPa, $\nu = 0.25$; b) $E_1 = 5.384 \cdot 10^4$ MPa, $E_2 = 1.795 \cdot 10^4$ MPa, $G_{12} = 0.863 \cdot 10^4$ MPa, $\nu_1 = 0.25$; c) $E_1 = 27.61 \cdot 10^4$ MPa, $E_2 = 2.761 \cdot 10^4$ MPa, $G_{12} = 1.035 \cdot 10^4$ MPa, $\nu_1 = 0.25$. The data for the isotropic material was obtained by taking the limit in the anisotropy parameters in the numerical solution.

Figures 1-4 show results of the calculations of K_1 and K_2 for the uniform tension of a half-plane by the forces $\sigma_y^\infty = 1$: curves 1 are for the isotropic material (case "a"), while curves 2 and 3 are for the orthotropic material (cases "b" and "c"); the change in the stress intensity factors is represented by solid (dashed) lines where the angle φ, formed by the principal anisotropy direction E_1 with the x axis is equal to 0 ($\pi/2$).

Figures 1 and 2 give values of K_i (i = 1, 2) for the left and right tips of a crack moving over a semicircular arc near the edge of the half-plane at Rh = 0.7 and $\ell = R/2$. The results obtained here for the isotropic material coincide with the data in [8]. Lines 1 pass between the curves for the orthotropic material at $\varphi = 0$ and $\pi/2$. The effect of anisotropy on the stress intensity factor may either decrease or increase, depending on the position of the crack relative to the edges of the half-plane and the loading axis. Anisotropy has the greatest effect at extreme values of K_i.

Figure 3 shows results of calculation of K_1 in relation to the angle α for an edge notch in the half-plane over an arc. Here, a is the length of the arc, $\ell = a/2$. The results of

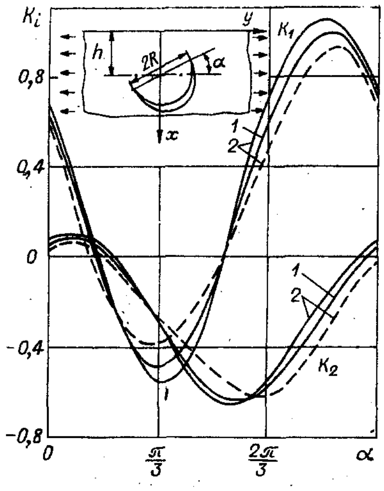

Fig. 1

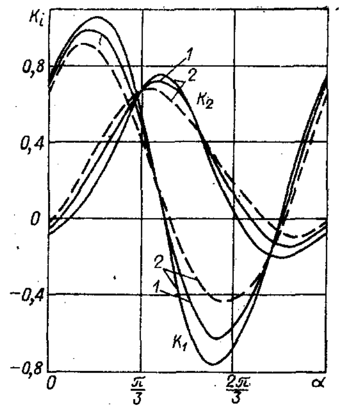

Fig. 2

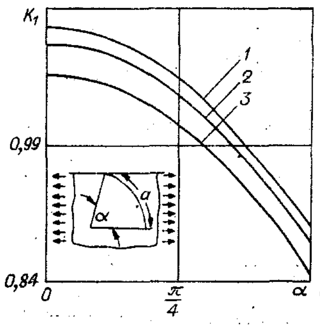

Fig. 3

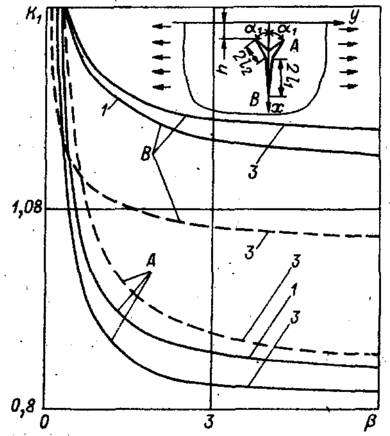

Fig. 4

TABLE 1

e	K_1					
	N					
	10	20	30	40	50	60
4	0.7493	0.7549	0.7549			
10	0.8213	0.8190	0.8188	0.8188		
16	0.8885	0.8782	0.8769	0.8768	0.8768	
28	0.9977	0.9828	0.9771	0.9765	0.9765	0.9765

the calculations for the isotropic material agree with the data in [9]. The value of K_1 decreases with an increase in the degree of anisotropy. At $\alpha \to 0$, we obtain values of K_1 at the tip of a straight edge notch $L = \{0 \leqslant x \leqslant a;\ y = 0\}$, which agree with the results in [10]. For the isotropic material at $\alpha \to 0$, $K_1 = 1.121$ for as many as 20 co-location points on the notch (N = 20), i.e., the relative error of the solution is no greater than 0.1% [1]. As for straight notches [10], the values of $K_1(\alpha)$ $(0 < \alpha < \pi/2)$ at $\varphi = 0$ and $\pi/2$ coincide.

Figure 4 shows the value of K_1 at the tips A and B of a branching crack (a main notch of length $2\ell_1$ normal to the plate edge dividing into two lateral notches of length $2\ell_2$ at angles $\alpha = \pi/6$) near the edge of a half-plane with $\ell_2/\ell_1 = 0.5$, $\ell = \ell_1/2$. The values of K_1 are plotted as a function of $\beta = h/\ell_1$. At $\beta \to \infty$ (the case of a branching crack in an infinite plate), the results of the calculations for an isotropic material coincide with well-known results in [1]. The divergence of K_1 at $\varphi = 0$ and $\pi/2$ increases with an increase in anisotropy, while the values of K_1 for the isotropic material occupy an intermediate position.

The calculated results demonstrated the good convergence of the algorithm. For comparison, Table 1 shows values of K_1 at $\alpha = 0$ and different values of $\varepsilon = R/h$ and N [see (2.2)] for the problem shown in Fig. 1. The convergence of the numerical solution deteriorates with the approach of the ends of the notch to the edge of the half-plane ($\varepsilon \to \infty$). For the problem shown in Fig. 3, the results of calculations of K_1 at $N \geq 20$ do not change in the first three significant digits.

For a cross-shaped crack $L = \bigcup_{s=1}^{4} L_s$ [four notches $L_s = \{t = \exp(i\pi s/2) l(\xi + 1);\ |\xi| < 1\}$ joined at the tips at angles $\alpha = \pi/2$] in an infinite isotropic plate subjected to the load $\sigma_x^\infty = \sigma_y^\infty = 1$, we have $K_1 = 0.86414$. The investigations [11, 12] obtained $K_1 = 0.86414$ and 0.86356, respectively, for this problem. All of the data presented is for 10 co-location points on each branch L_s (s = 1, 4).

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